freedom. Where a substructure is relatively stiff in a particular direction, it may be desirable to define the substructure in terms of less elements for motion in this direction, and more lumped masses for motion in the other more significant directions. In this way, the structure can be represented three-dimensionally with a view toward economizing the number of mass-inertia elements employed.

References

[1] Dugundji, J., "On the calculation of natural modes of free-free structures," J. Aerospace Sci. **28,** 164–166 (1961).

An Estimation Lemma for Laminar Compressible Boundary-Layer Velocity Profiles

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Knowledge of the velocity and temperature fields in a laminar steady two-dimensional boundary-layer requires the solution of a pair of coupled parabolic equations describing the flow of momentum and energy. It is much easier to obtain the same information for an incompressible boundary layer, since there the momentum equation is independent of the solution of the energy equation. The present note describes the use of the solution of a related incompressible problem to estimate the solution of the momentum equation of the original compressible problem. The fluid considered is an ideal gas, the viscosity of which is directly proportional to its temperature.

The compressible flow equations used here are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\rho_1}{\rho}u_1u_1'(x) - \nu_1\frac{\rho_\infty}{\rho}\frac{\partial}{\partial y}\left(\frac{\rho_\infty}{\rho}\frac{\partial u}{\partial y}\right) = 0 \quad (1)$$

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \quad (2)$$

where

$$\nu_1 = (\mu_\infty/\rho_\infty)(P_1/P_\infty) \quad (3)$$

Here the subscript 1 denotes the limiting value of the dependent variable as y becomes very large, and the subscript ∞ denotes a reference value that will be determined later. The rest of the notation is conventional.

The set of equations would be completed by energy and state equations. Boundary conditions are specified at some upstream value of x, say at $x = 0$, at $y = 0$ (denoted by the subscript w), and as y becomes very large (denoted by the subscript 1). T_w, ρ_w, T_1, ρ_1, P_1, and u_1 are, in general, functions of x. The problem is now correctly posed for the four variables u, v, ρ, and T. The pressure $P(x, y)$ is assumed to be equal to the external flow value $P_1(x)$ for all y.

The energy equation is not used in the approximate method presented here. Instead, we assume only that the velocity and temperature increase monotonically with y from the wall values to the external flow values. The restriction to monotonic temperature profiles means that, for a given pressure field, Prandtl number, and ratio of wall temperature to external flow temperature, there is an upper bound on the external flow Mach number. For example, in the case of a flat plate in a gas of unit Prandtl number, the Crocco integral of the energy equation gives

$$M_1 \leq \left[\frac{2}{\gamma - 1}\left(1 - \frac{T_w}{T_1}\right)\right]^{1/2}$$

for the temperature profile to be monotonic. Thus if $\gamma = 1.4$ and the wall temperature is $\frac{4}{5}$ of the external flow temperature, the Mach number can be no greater than 1, whereas if the wall temperature is $\frac{1}{5}$ of the external flow temperature, the Mach number can be as large as 2. For more general pressure fields and Prandtl numbers the assumption of monotonic temperature profiles will also imply an analogous limitation to moderate Mach numbers. Such a restriction to transonic or low supersonic Mach numbers would be necessary in any case if the linear viscosity-temperature law is expected to be a realistic approximation to the behavior of a diatomic gas. We also restrict our statements to cases where the external flow velocity is a nondecreasing function of x.

The problem is simplified by the introduction of the related independent variables x^* and y^*, which are given by the Howarth-Dorodnitsyn transformation

$$x^*(x) = x \quad (4)$$

$$y^*(x, y) = \frac{1}{\rho_\infty}\int_0^y \rho(x, t)dt \quad (5)$$

The transformed dependent variables will also be denoted by the superscript (*). For example,

$$u^*(x^*, y^*) = u(x, y) \quad (6)$$

Instead of Eqs. (1) and (2), we consider the pair of equations

$$u^*\frac{\partial u^*}{\partial x^*} + w^*\frac{\partial u^*}{\partial y^*} - \frac{\rho_1}{\rho^*}u_1u_1'(x) - \nu_1\frac{\partial^2 u^*}{\partial y^{*2}} = 0 \quad (7)$$

$$(\partial u^*/\partial x^*) + (\partial w^*/\partial y^*) = 0 \quad (8)$$

Note that w^* is merely a function defined by Eqs. (7) and (8); no statements are made about its relation to v or v^*.

In order to obtain the estimation lemma we will need

$$y^*(x, y) \leq y \quad (9)$$

at fixed x and y. Examination of Eq. (5) shows that this is satisfied if

$$\rho/\rho_\infty \leq 1 \quad (10)$$

everywhere in the boundary layer. This enables us to choose the reference values. We let T_∞ be the value of T_w at that x where P_1/T_w is greatest. Then P_∞ is P_1 at the same x, ρ_∞ is found from the ideal-gas law, and μ_∞ is found from a table of gas properties. This choice of T_∞ and P_∞ satisfies Eq. (10), which satisfies Eq. (9).

The earlier assumption that u is a monotonically increasing function of y implies that u^* is a monotonically increasing function of y^*, since

$$\frac{\partial u}{\partial y} = \frac{\partial y^*}{\partial y}\frac{\partial u^*}{\partial y^*} = \frac{\rho}{\rho_\infty}\frac{\partial u^*}{\partial y^*}$$

or

$$\frac{\partial u^*}{\partial y^*} = \frac{\rho_\infty}{\rho}\frac{\partial u}{\partial y}$$

and the right-hand side of this last equation is positive. Because of this result and Eq. (9), we see that Eq. (6) implies that

$$u(x, y) \leq u^*[x^*(x), y^*(x, y)] \quad (11)$$

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at fixed x and y. Therefore, the solution u^* of Eqs. (7) and (8) forms an upper bound for the solution u of Eqs. (1) and (2) when compared at the same values of x and y.

We cannot solve Eqs. (7) and (8) since ρ^* is unknown, but we can find approximate solutions by replacing ρ^* by ρ_1. These solutions will be called $\bar{u}^*$ and $\bar{w}^*$; they satisfy

$$\bar{u}^* \frac{\partial \bar{u}^*}{\partial x^*} + \bar{w}^* \frac{\partial \bar{u}^*}{\partial y^*} - u_1 u_1'(x) - \nu_1 \frac{\partial^2 \bar{u}^*}{\partial y^{*2}} = 0 \qquad (12)$$

$$(\partial \bar{u}^*/\partial x^*) + (\partial \bar{w}^*/\partial y^*) = 0 \qquad (13)$$

and will be assumed to be known. The relation between $\bar{u}^*$ and u^* is specified in a lemma developed by Nickel,[1] the result of which is as follows. Let $\bar{u}^*$ and u^* be solutions defined in an open region G_1, consisting of $x_1^* < x^* < x_2^*$, $0 < y < \infty$, of the pairs of equations

$$\bar{u}^* \frac{\partial \bar{u}^*}{\partial x^*} + \bar{w}^* \frac{\partial \bar{u}^*}{\partial y^*} - UU'(x^*) - \nu \frac{\partial^2 \bar{u}^*}{\partial y^{*2}} = 0$$

$$\frac{\partial \bar{u}^*}{\partial x^*} + \frac{\partial \bar{w}^*}{\partial y^*} = 0$$

and

$$u^* \frac{\partial u^*}{\partial x^*} + w^* \frac{\partial u^*}{\partial y^*} - UU'(x^*) - \nu \frac{\partial^2 u^*}{\partial y^{*2}} \underset{(\geq)}{\leq} 0$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial y^*} = 0$$

respectively. If $\bar{u}^*$ and u^* are monotonically increasing functions of y^*, which satisfy the boundary conditions

$$\bar{u}^*(x_1^*, y^*) = u^*(x_1^*, y^*) = \tilde{u}^*(y^*) \qquad \text{for } 0 \leq y^* < \infty$$

$$\bar{u}^*(x^*, 0) = u^*(x^*, 0) = 0 \qquad \text{for } x_1^* \leq x^* < x_2^*$$

and

$$\lim_{y^* \to \infty} \bar{u}^*(x^*, y^*) = \lim_{y^* \to \infty} u^*(x^*, y^*) = U(x^*) \qquad \text{for } x_1^* \leq x^* < x_2^*$$

then

$$u^*(x^*, y^*) \underset{(\geq)}{\leq} \bar{u}^*(x^*, y^*)$$

at fixed values of x^* and y^* everywhere in G.

The problem considered in the present note differs slightly from the one considered by Nickel, in that $u_1(x)$ appears in place of $U(x^*)$, and $\nu_1(x)$ in place of ν. The two variables x and x^* are related by Eq. (4), and Nickel's lemma remains valid when the kinematic viscosity has a specified x^* dependence, so these differences do not affect its conclusion. If we write Eq. (7) as

$$u^* \frac{\partial u^*}{\partial x^*} + w^* \frac{\partial u^*}{\partial y^*} - u_1 u_1'(x) - \nu_1 \frac{\partial^2 u^*}{\partial y^{*2}} = \left(\frac{\rho_1}{\rho^*} - 1\right) u_1 u_1'(x) \qquad (14)$$

we see that, because the temperature profile is monotonic and the external flow velocity is a nondecreasing function of x, the right-hand side of Eq. (14) is never positive. Since $\bar{u}^*$ is the solution of Eq. (14) with the right-hand side replaced by zero, we conclude that

$$u^*(x^*, y^*) \leq \bar{u}^*(x^*, y^*) \qquad (15)$$

by Nickel's lemma. Combining this result with that of Eq. (11), we have

$$u(x, y) \leq \bar{u}^*[x^*(x), y^*(x, y)] \qquad (16)$$

at fixed values of x and y.

The result of the note can be summarized as follows. Say that we have a compressible boundary-layer problem described by Eqs. (1–3) with prescribed boundary conditions on the velocity and with conditions such that the assumptions of monotonic velocity and temperature profiles and the nondecreasing external flow velocity appear reasonable. Then the solution of the related incompressible problem, which has the same x dependent kinematic viscosity and the same boundary conditions on the velocity, gives an upper bound to the velocity profile of the compressible problem.

The result has two immediate generalizations. First, it is reversible in the sense that if the wall is hotter than the external flow we could show that under comparable restrictions the incompressible solution would be a lower bound. Second, the kinematic viscosity can be given an arbitrary known x dependence, such as that described by Stewartson[2] and referred to as the Chapman viscosity law. This can be used to give a more realistic viscosity-temperature relation in the neighborhood of the wall.

References

[1] Nickel, K., "Eine Einfache Abschätzung für Grenzschichten," Ing. Arch. **31**, 85–100 (1962).

[2] Stewartson, K., *The Theory of Laminar Boundary Layers in Compressible Fluids* (Oxford University Press, London, 1964), p. 5.

A New Technique for the Direct Calculation of Blunt-Body Flow Fields

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Nomenclature

h = $(1 - V^2)^{1/(\gamma-1)}\,v$
H = $\rho v^2 + Kp$
g = $\rho u^2 + Kp$
G = $r^j g/R + j(1 + n/R)\cos\theta Kp$
j = 0, 1, two-dimensional problem and axisymmetric problem, respectively
K = $(\gamma - 1)/2\gamma$
p = pressure
s, n = body oriented curvilinear coordinates
t = $(1 - V^2)^{1/(\gamma-1)}u$
u = velocity component in the s direction
v = velocity component in the n direction
V^2 = $u^2 + v^2$
ξ = shock angle
Z = ρuv
γ = ratio of specific heats
δ = shock distance in the n direction
θ = surface angle of the body
ρ = density

Subscripts

0 = condition on the body surface
δ = condition immediately behind the shock wave

Superscripts

$(\,^-)$ = initial condition on the stagnation axis

All velocities are nondimensionalized by the maximum possible velocity, density by freestream stagnation density, and pressure by freestream stagnation pressure.

Introduction

THE direct integral method for calculating the inviscid hypersonic flow over a blunt body, developed by Belotserkovskii and based on a general numerical method for solving nonlinear hydrodynamic differential equations by

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